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GENERAL AIRFOIL THEORY*

By H. G. Küssner

On the assumption of infinitely small disturbances the author develops a generalized integral equation of airfoil theory which is applicable to any motion and compressible fluid. Successive specializations yield various simpler integral equations, such as Possio's, Birnbaum's, and Prandtl's integral equations, as well as new ones for the wing of infinite span with periodic downwash distribution and for the oscillating wing with high aspect ratio. Lastly, several solutions and methods for solving these integral equations are given.

INTRODUCTION

There are a number of airfoil theories which hold true in two or three dimensions, are stationary or nonstationary, and allow or disallow for the compressibility of air. All these theories have one thing in common: They are, strictly speaking, valid only for infinitely small disturbances; hence the airfoil must be assumed as infinitely thin and with infinitely small deflections from a regulating surface , the generating line of which is parallel to the direction of flight. Then the regulating surface itself can be approximately considered as the place of the wing and the area of discontinuity emanating from its trailing edge. Up to the present time, a plane has been commonly chosen as a regulating surface, but this restriction is not necessary.

Following the temporary interest attaching to the vortex theory, the introduction of Prandtl's acceleration potential made the old potential theory applicable to airfoil theory. The particular advantage of this method over

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the vortex method is that the compressibility of air can be taken into account. By this method the wing is replaced by an arrangement of acoustic "radiators." The sole essential restriction of this theory consists in the assumption of moderate fields of sound, that is, small interferences. Then the classical wave equation

$$\Box \Phi \equiv \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} - \frac{\partial^2 \Phi}{c^2 \partial t^2} = 0 \tag{1}$$

is applicable for the velocity potential and the sound pressure of a quiescent source distribution. Their solutions have already been explored in all directions. Its application to the moving airfoil is achieved with the aid of the well-known Lorentz transformation, the sole invariant being the speed of radiation c, which, in the case in question, is equal to the velocity of sound.

The setting-up of the integral equations of the airfoil theory is a preliminary task, which is definitely achieved by the subsequent expositions. But this preliminary work alone accomplishes little without attacking the purely mathematical main problem, namely, the solution of these integral equations without entering into new discussions every time regarding the method of derivation and its physical significance.

2. THE VELOCITY POTENTIAL OF THE ELEMENTARY RADIATOR

The wave equation $\Box \Phi = 0$ has, as is known, a very simple solution for a spherical wave that spreads out radially at speed c from its source. The solution reads:

$$\Phi_o = \frac{1}{r} f\left(t - \frac{r}{c}\right) \tag{2}$$

where r denotes the radius, t the time, and f an arbitrary function. Such a spherical wave is produced by an elementary radiator of zero order, which represents a simple point source. The velocity potential of radiators of higher order follows from (2) by partial derivation along any coordinate directions.

The airfoil which is to manifest a pressure difference between its two surfaces is best replaced by a superposition of radiators of the first order (so-called doublets) whose axes are normal to the airfoil. With n denoting the direction of the normals, the potential of a doublet of the superposition is

$$\Phi_1 = \frac{\partial \Phi_o}{\partial n} \tag{3}$$

The sound pressure of the field of sound is

$$p = - \rho \frac{\partial \Phi}{\partial t} \tag{4}$$

hence it satisfies equations $\Box p = 0$ also because $\rho \sim$ constant for very small disturbances.

3. THE LORENTZ TRANSFORMATION

The arguments so far have dealt with the radiator at rest at infinity. To reach the pressure field of the radiator moving at constant speed $v < c$ past the X axis, the Lorentz transformation is resorted to. With the prime ' indicating the transformed system, the transformation formulas for a motion along the negative x axis of the quiescent reference system read (reference 1):

$$x = \frac{X' + vt'}{\sqrt{1 - v^2/c^2}}; \quad y = y'; \quad z = z'; \quad t = \frac{t' + v X'/c^2}{\sqrt{1 - v^2/c^2}} \tag{5}$$

On passing to the coordinate $x' = X' + vt'$ in the entrained reference system and posing Mach's number $\beta = v/c$, equation (5) gives:

$$x = \frac{x'}{\sqrt{1-\beta^2}}; \quad y = y'; \quad z = z'; \quad t = t' \sqrt{1 - \beta^2} + \frac{x' \beta}{c \sqrt{1 - \beta^2}} \tag{6}$$

According to (6), the change to the moving reference system entails the following substitutions:

$$r^2 = \frac{x'^2}{1 - \beta^2} + y'^2 + z'^2 \tag{7}$$

$$-\frac{r}{c} = \sqrt{1-\beta^2}\left[t' + \frac{x'\beta}{c(1-\beta^2)} - \frac{\sqrt{x'^2 + (1-\beta^2)(y'^2+z'^2)}}{c(1-\beta^2)}\right]$$

Inasmuch as the airfoil is to be situated on a regulating surface with generating line parallel to flow direction, that is, along x direction, the Lorentz transformation does not alter the direction of the normals n.

Later, the knowledge of the sound pressure of a moving doublet is necessary. According to equations (2), (3), and (4), the sound pressure for the quiescent doublet, the sequence of the differentiations being changed, is:

$$p = -\rho \frac{\partial}{\partial n}\left(\frac{1}{r}\frac{\partial}{\partial t} f\left(t - \frac{r}{c}\right)\right) \tag{8}$$

Since f is an arbitrary function, $\frac{\partial f}{\partial t}$ can be replaced by another arbitrary function Υ; and constant factors such as $\sqrt{1-\beta^2}$ can then be omitted or added. After completing the Lorentz transformation, equations (7) and (8) give for the moving doublet the pressure

$$p(x,y,z,t) = -\frac{\rho v}{4\pi}\frac{\partial}{\partial n}\frac{\Upsilon\left(t + \frac{x\beta}{c(1-\beta^2)} - \frac{\sqrt{x^2 + (1-\beta^2)(y^2+z^2)}}{c(1-\beta^2)}\right)}{\sqrt{x^2 + (1-\beta^2)(y^2+z^2)}} \tag{9}$$

where the transformation prime can then be omitted.

4. The Acceleration Potential

According to Prandtl (reference 2) a new approach to the airfoil theory is afforded by proceeding from the field of the acceleration vector

$$\underline{b} = \frac{D\underline{w}}{dt} = \frac{\partial \underline{w}}{\partial t} + \underline{w} \circ \nabla \underline{w}$$

where $\underline{w}$ is the velocity vector.

Assuming frictionless, homogeneous fluid gives the Euler equation:

$$\underline{b} + \frac{1}{\rho}\,\text{grad } p = \underline{b} + \text{grad} \int \frac{dp}{\rho} = 0 \tag{10}$$

Hence $\underline{b}$ is also a gradient, and an "acceleration potential" φ can be introduced, so that $\underline{b} = \text{grad } \varphi$. Then integration of (10) gives

$$\varphi + \int \frac{dp}{\rho} = f(t) \tag{11}$$

If the flow is undisturbed at infinity, then $f(t) =$ const. If, in addition, it is restricted to small disturbances, then $\rho \sim$ const. and the acceleration

$$\varphi = -\frac{p}{\rho} + \text{const} \tag{12}$$

follows from (11).

On the airfoil, pressure p, and hence φ, manifests a jump (or discontinuity). At any point outside the lifting surface the value p, and hence φ, is continuous for $v < c$.

The acceleration potential φ can be approximated, in the case of compressible fluid, from a doublet superposition. This procedure would be rigorously executed for incompressible fluid, but for ~~in~~compressible fluid the pressure amplitudes would have to be small. This superposition can also be assumed on the regulating surface instead of on the lifting surface, provided the interferences are very small.

Through the superimposed doublets, the intensity of which at any point corresponds to the magnitude of the pressure jump on the lifting surface, the acceleration potential φ is defined and follows from (9) and (12). Then the velocity field must be defined from φ. On the previous assumption that the flying speed v is constant and that the lifting surface moves along the negative x axis, the acceleration vector is

$$\underline{b} = \frac{\partial \underline{w}}{\partial t} + v\,\frac{\partial \underline{w}}{\partial x} \tag{13}$$

the terms of the second order being ignored.

Posing $\underline{b} = \text{grad } \varphi$, $\underline{w} = \text{grad } \Phi$, followed by integration (13), gives

$$\varphi = \frac{\partial\Phi}{\partial t} + v\,\frac{\partial\Phi}{\partial x} \tag{14}$$

Given $\varphi(x,y,z,t)$, the desired acceleration potential finally follows from (14) at

$$\Phi(x,y,z,t) = \frac{1}{v}\int_{-\infty}^{x}\varphi\left(x',y,z,t - \frac{x - x'}{v}\right)dx' \tag{15}$$

One particular advantage ofthis method of representation is that the so-called area of discontinuity behind the airfoil plays no part because, though it manifests a velocity jump, it has no pressure jump and hence no discontinuity in φ.

An element $d\sigma$ of the surface covered with doublets furnishes, according to (9), (12), and (15), the potential proportion

$$d\Phi = \frac{d\sigma}{4\pi}\int_{-\infty}^{x}\frac{\partial}{\partial n}\,\frac{\gamma\left(t - \frac{x-x'}{v} + \frac{x'\beta}{c(1-\beta^2)} - \frac{\sqrt{x'^2+(1-\beta^2)(y^2+z^2)}}{c(1-\beta^2)}\right)}{\sqrt{x'^2 + (1-\beta^2)(y^2+z^2)}}\,dx' \tag{16}$$

The analyzed element $d\sigma$ lies in the zero point of our coordinate system, that is, at $x, y, z = 0$.

5. THE GENERAL INTEGRAL EQUATION OF AIRFOIL THEORY

The location of our elementary radiator is now shifted to any point of the airfoil with the coordinates ξ, $y(\eta)$, $z(\eta)$. Coordinate ξ is measured parallel to the generating line of the regulating surface; hence along the x axis. Coordinate η is so chosen that, after development of the regulating surface in a plane, ξ and η form a system of Cartesian coordinates, and the substitution

$$\begin{array}{ccc} x & y & z \\ x - \xi & y - y(\eta) & z - z(\eta) \end{array} \tag{17}$$

must be effected in (16).

The element of the surface of the airfoil is $d\sigma = d\xi d\eta$. Ordinarily γ will still be a function of ξ and

η because of the variation of doublet intensity on the airfoil. The differentiation along the direction of the normals can be divided in

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial y}\frac{dy}{dn} + \frac{\partial}{\partial z}\frac{dz}{dn} = \sin\alpha(\eta)\frac{\partial}{\partial y} + \cos\alpha(\eta)\frac{\partial}{\partial z} \tag{18}$$

where angle $\alpha(\eta)$ between the direction of the normals and the z axis is a function of the curved coordinate η. Posting (17) and (18) in (16), followed by integration over the airfoil, gives the complete term of the velocity potential of the airfoil at:

$$\Phi(x,y,z,t) = \frac{1}{4\pi}\iint_{F1}\int_{-\infty}^{x'=x-\xi} dx'd\xi d\eta\left(\sin\alpha(\eta)\frac{\partial}{\partial y} + \cos\alpha(\eta)\frac{\partial}{\partial z}\right)$$

$$\frac{\gamma\left(\xi,\eta,t+\dfrac{x'-x+\xi}{v}+\dfrac{x'\beta}{c(1-\beta^2)}-\dfrac{\sqrt{x'^2+(1-\beta^2)[(y-y(\eta))^2+(z-z(\eta))^2]}}{c(1-\beta^2)}\right)}{\sqrt{x'^2+(1-\beta^2)[(y-y(\eta))^2+(z-z(\eta))^2]}} \tag{19}$$

On the other hand, the pressure jump Π on the airfoil is proportional to the intensity of the doublets. The constant factors in (9) are precisely so chosen that

$$\Pi(\xi,\eta,t) = \rho v\gamma(\xi,\eta,t) \tag{20}$$

This conforms to usual practice: lift, positive upward, downwash, positive downward. Literature at times quotes the more abstract: downwash, positive upward, in which case the prefix of the right-hand side of (20) must be reversed.

Equations (19) and (20) represent the most general integral equation of the airfoil theory for small disturbances that can be used for computing the pressure jump Φ for a given downwash. Equation (19) represents a boundary value problem. Its solution rests on the fact that the downwash on the airfoil itself is given by the type of motion and form change of the airfoil in first approximation. Assume that $n = n(x,\eta,t)$ is a small deflection of the airfoil from the regulating surface in direction of the normals. Then the downwash on the airfoil - the terms of the second order being neglected - is

$$w(x,y(\eta),z(\eta),t) = \frac{\partial n}{\partial t} + v\,\frac{\partial n}{\partial x} \qquad (21)$$

When solving (19), the Kutta flow-off condition must be met. It is met when $\gamma = 0$ on the trailing edge. A frequently employed formula, which corresponds to this condition, is:

$$\gamma(\xi,\eta,t) = g(\eta,t)\left[a_o \cot\frac{\theta}{2} + \sum_1^\infty a_n \sin n\,\theta\right]$$

where

$$\cos\theta = -\frac{\xi}{l}$$

and $l = l(\eta)$, half the wing chord. For special purposes, such as airfoils with circular contour, for instance, combinations of spherical functions of the first and second kind are used, which also satisfy the Kutta condition.

It may be mentioned that the vortex method also arrives at the quantity γ defined by (20). It indicates then the density of the bound vortices. This quantity γ has, however, a much more general significance than doublet intensity indication since the vortex concept is restricted to incompressible fluid, while (19) and (20) apply equally to compressible fluids.

6. SPECIAL FORMS OF INTEGRAL EQUATION (19)

The general integral equation (19) can be specialized in several directions.

a) For solving the boundary value problem, the downwash on the airfoil itself is used; that is, for points of origin with the coordinate $y = y(\eta_1)$; $z = z(\eta_1)$. Since solution (2) of the wave equation upon which our integral bases applies only outside the singular point $r = 0$ in view of the linearization, one may not integrate through the doublets because then the integral over x' becomes divergent. Hence the downwash on the airfoil itself can be obtained only by a limit transition from (19). The method nearest at hand consists in carrying out the integration with respect to two surfaces at distance $+\epsilon$ and $-\epsilon$ from the airfoil, in forming the average value and proceeding to $\epsilon \longrightarrow 0$. This method, applied to (19), leads to complicated formulas. On the other hand, the same result is achieved if, after the differentiations, $y = y(\eta_1)$ and $z = z(\eta_1)$, are formerly written in (19) and the divergent integral is then so transformed by partial integration that Cauchy's principal value can be formed. This change makes the formula longer and, in general, less comprehensive. So in the following we confine ourselves in most cases to the statement of the simpler divergent integral while tacitly presuming the further transformation in the Cauchy principal value. If the integration with respect to x' is made numerically, the divergence can be avoided quite simply, by either integrating from $x - \xi$ to $-\infty$ or from $+\infty$ to $x - \xi$, depending upon whether $x - \xi < 0$ or > 0. For in any event the downwash must disappear at infinity. In the divergent method of writing the downwash on the airfoil ultimately assumes the form:

$$\frac{\partial\Phi}{\partial n} = w(x,y(\eta_1),z(\eta_1),t) = \frac{1}{4\pi}\iint_{F1}\int_{-\infty}^{x'=x-\xi} dx'\,d\xi\,d\eta \left[\left(\sin\alpha(\eta)\frac{\partial}{\partial y} + \cos\alpha(\eta)\frac{\partial}{\partial z}\right)^{2} \frac{\gamma\left(\xi,\eta,t+\frac{\xi-x}{v}+\frac{x'}{v(1-\beta^2)}-\frac{\sqrt{x'^2+(1-\beta^2)\left[(y-y(\eta))^2+(z-z(\eta))^2\right]}}{c(1-\beta^2)}\right)}{\sqrt{x'^2+(1-\beta^2)\left[(y-y(\eta))^2+(z-z(\eta))^2\right]}}\right]_{\substack{y=y(\eta_1)\\ z=z(\eta_1)}} \quad (2$$

b) It can be assumed that the regulating surface is a plane, say, the xy plane, for instance. Airplane wings are usually flat structures. Then $\alpha = 0$, $y(\eta) = \eta$, and $z(\eta) = 0$. With specialiaztion a) $z = 0$ also. This assumption itself effects a substantial simplification in the equation form and has been practically always introduced in the airfoil theory so far.

c) Some special assumptions regarding the type of time rate of change can be made. The linearity of the integral equation affords especially simple forms on the assumption of harmonic processes with respect to time; hence harmonic oscillations of the airfoil. The preeminent importance of the harmonic solutions of the wave equation in physics is an established fact. Once harmonic solutions of the integral equation are known, solutions for any time rate of change of downwash can be found by superposition whereby the ~~reversed~~ inverse Laplace transformation plays a prominent part. Proceeding to the limiting case of very small oscillation and putting $\gamma = \gamma(\xi,\eta)$ results in the stationary form of the integral equation. Encumbered with further restrictions, this special case has been treated most so far.

d) Simplified assumptions regarding the shape of the airfoil can be made. Allowing the span of the airfoil to increase to infinity and assuming the downwash to be independent of η gives the same flow process in all the planes η = const, so that the integration with respect to η can be effected. The form of the integral equation so obtained represents a two-dimensional flow process. This form is of particular interest because it is solved for any downwash function under the restriction $\beta = 0$ and for stationary flow also with $\beta \neq 0$. The significance of this solution in the so-called vortex filament theory will be explained elsewhere.

On an airfoil of constant chord and infinite span both the downwash distribution periodic in η

$$w(\xi,\eta,t) = w_0(\xi,t) \exp i\mu\eta$$

and that independent of η can be taken into consideration. This defines a γ distribution periodic in η. The integration with respect to η can be effected in various cases. From these harmonic solutions superposition affords solutions for any finite airfoil of constant chord, in motion. The method is therefore applicable only to airfoils having a parallelogram (especially a rectangle) as contour and pressures that no flow passes around the lateral edges of the parallelogram.

The ultimate aim of the simplifications is the change of the surface integral in (19) to a line integral, which naturally is more promising for a solution of the integral equation. This aim can also be reached by assuming airfoil contours representing coordinate lines in simple systems of curvilinear orthogonal coördinates. On airfoils of elliptic plan form the confocal elliptic coordinates permit integration with the aid of the Lamé functions. For circular contour there are the spherical functions of the first and second type. Examples can be found in the reports by Kinner (reference 3), Krienes (reference 4), and Schade (reference 5).

e) It is readily apparent from (19) that the assumption of incompressible fluid affords a substantial simplification in the form of the integral equation because

$\beta = 0$ and $c = \infty$. One exception is the stationary flow, where, moreover, the complicated argument of γ, which starts with t, disappears. Allowance for the compressibility then consists merely in the reduction of the x coordinate in the ratio $1:\sqrt{1 - \beta^2}$, the so-called Prandtl contraction, which represents the exact counterpart of the Lorentz contraction in the special relativity theory. The downwash therefore should be computed for an airfoil with less chord, which thus affords by given pressure jump a smaller downwash and, conversely, by given downwash a higher pressure jump than for $\beta = 0$.

In the case of stationary flow it forthwith affords solutions $\beta \neq 0$, as soon as solutions for $\beta = 0$ are available.

The oscillating airfoil for $0 < \beta < 1$ has been treated in only one report (reference 7), and then, as a two-dimensional problem. All other investigations of nonstationary airfoil motions are restricted to the assumption of incompressible fluid. The reason for this is to be found in the fact that the general case $\beta \neq 0$ was not approachable by the vortex method solely employed heretofore until Prandtl introduced the acceleration potential.

7. THE AIRFOIL OF INFINITE SPAN

The integration with respect to η on an airfoil of constant chord and infinite span in (19) is dictated by a special assumption for the relation of function γ with time τ. We put

$$\gamma(\xi, \tau) = g(\xi) \exp i \nu \tau$$

that is, study the harmonic oscillations of the airfoil. This assumption implies no loss in general validity since any others can be built up from the harmonic solutions by superposition. As the ultimate result a closed solution even can be indicated which is applicable to any downwash function without resorting to harmonic analysis. With the specializations a), b), and c) we obtain from (22) by derivation according to z the downwash along the z axis at:

$$w(x,y,z,t) = \frac{1}{4\pi} \int\limits_{\xi=-l}^{+l} \int\limits_{\eta=-\infty}^{+\infty} \int\limits_{-\infty}^{x'=x-\xi} d\,x'\, d\,\eta\, d\,\xi\ \gamma(\xi,t)$$

$$\cdot \frac{\partial^2}{\partial z^2} \frac{\exp i\,\nu \left[\frac{\xi-x}{v} + \frac{x'}{v(1-\beta^2)} - \frac{\sqrt{x'^2+(1-\beta^2)((y-\eta)^2+z^2)}}{c(1-\beta^2)}\right]}{\sqrt{x'^2 + (1-\beta^2)((y-\eta)^2 + z^2)}} \tag{23}$$

When the new variable

$$u^2 = \frac{(1 - \beta^2)(y - n)^2}{x'^2 + (1 - \beta^2)z^2} + 1$$

is introduced into (23), bearing in mind that the part integrals with respect to η between the limits $+\infty$ to y and y to $-\infty$ must be of identical magnitude, the integration with respect to η can be effected with the cylindrical function (reference 6)

$$\int_1^\infty \exp(-ixu)\, \frac{du}{\sqrt{u^2 - 1}} = -\frac{i\pi}{2} H_0^{(2)}(x) \tag{24}$$

and (23) and (24) give the Possio integral equation

$$w(x,z,t) = \frac{-i}{4\sqrt{1-\beta^2}} \int\limits_{\xi=-l}^{+l} \int\limits_{-\infty}^{x'=x-\xi} dx'd\xi\,\gamma(\xi,t) \exp \frac{i\nu}{v}\left[\xi - x\right.$$

$$\left. + \frac{x'}{1-\beta^2}\right] \frac{\partial^2}{\partial z^2} H_0^{(2)} \left(\frac{\nu\sqrt{x'^2 + (1-\beta^2)\ z^2}}{c(1-\beta^2)}\right) \tag{25}$$

Possio (reference 7) does not give this equation explicitly, but merely its kernel in the complex real form in order to compute numerical approximate solutions. $H_0^{(2)}$ denotes Hankel's cylindrical function of the second type, l half the airfoil chord. The origin of the coordinate system is at airfoil center.

After differentiation and insertion of $z = 0$, equation (25) gives the downwash at

$$w(x,0,t) = \frac{\sqrt{1 - \beta^2}}{2\pi} \int_{-l}^{+l} \gamma(\xi,t) \exp \frac{i\nu}{v} (\xi - x)$$

$$R\left(\beta,\ \frac{\nu}{v}\frac{x - \xi}{1 - \beta^2}\right) \frac{d\xi}{x - \xi} \tag{26}$$

wherein

$$R(\beta,y) = \frac{i\pi}{2}\ \beta\ y \int_{-\infty}^{y} \exp\ i\ u\ H_1^{(2)}(\beta\,|u|)\ \frac{du}{|u|}$$

$$= -\frac{i\pi}{2}\ \beta\ y \int_{y}^{\infty} \exp\ i\,u\,H_1^{(2)}(\beta\ u)\ \frac{du}{u} \quad \text{for} \quad y > 0$$

$$= \frac{i\pi}{2}\ \beta\ y \int_{-y}^{\infty} \exp\ (-iu) H_1^{(2)}(\beta\ u)\ \frac{du}{u} \quad \text{for} \quad y < 0 \tag{27}$$

The absolute value $|u|$ appears in (27) and farther on because the argument has the significance of a radius. With

$$\lim_{x \to 0} x\ H_1^{(2)}(x) = \frac{2\ i}{\pi}$$

it affords for the function R the special values

$$R(\beta,0) = 1 \tag{28}$$

$$R(0,y) = -y \int_{-\infty}^{y} \exp\ i\ u\ \frac{du}{u^2} \tag{29}$$

In stationary flow $\nu = 0$ and (26) and (28) afford

$$w(x,0,t) = \frac{\sqrt{1 - \beta^2}}{2\pi} \int_{-l}^{+l} \gamma(\xi)\ \frac{d\xi}{x - \xi} \tag{30}$$

The root factor represents Prandtl's law, according to which the compressibility raises the pressure jump in the ratio $1:\sqrt{1 - \beta^2}$. Integral equation (30) was originally solved closed by Munk (reference 8) for $\beta = 0$.

In compressible fluid $\beta = 0$ and (26) and (29) give, with another integration variable $x'' = x - \frac{v}{\nu} u$, the Birnbaum integral equation

$$w(x,0,t) = -\frac{1}{2\pi} \int_{\xi=-l}^{+l} \int_{x''=\xi}^{\infty} \gamma(\xi,t) \exp \frac{i\nu}{v}(\xi - x'') \frac{dx'' d\xi}{(x-x'')^2} \quad (31)$$

Birnbaum (reference (9)) writes his equation in the convergent form following from (31) after partial integration and forming Cauchy's principal value, whereby the sequence of the integrations can be changed.

Equations (30) and (31) could also be deduced from (23) direct, by putting $\nu = 0$ and $c = \infty$ before integrating.

Birnbaum's equation was solved closed by Küssner and Schwarz (reference 10) with allowance for Kutta's flow-off condition. For more convenient representation of the solution it is expedient to introduce the new variables

$$x = -l \cos \theta ; \quad \xi = -l \cos \vartheta$$

and the parameter

$$\omega = i \bar{\omega} = \frac{i\nu l}{v}$$

Then the solution, on the premise of harmonic time function, reads

$$\gamma(\theta,t) = \frac{1}{\pi} \int_0^{\pi} \Big\{ [1+\cos\vartheta + T(-i\omega)(1-\cos\vartheta)] \cot\frac{\theta}{2}$$

$$+ \frac{2 \sin\theta}{\cos\theta - \cos\vartheta} + \omega \sin\vartheta \ln \frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)} \Big\} w(\vartheta,t)\, d\vartheta \quad (32)$$

$$T(\bar{\omega}) = \frac{H_0^{(2)}(\bar{\omega}) + i\, H_1^{(2)}(\bar{\omega})}{-H_0^{(2)}(\bar{\omega}) + i\, H_1^{(2)}(\bar{\omega})} \quad (33)$$

The complex function T can be computed with the aid of tabulated functions and is given in table 1.

This result has been applied by the author (reference 11) with the aid of the superposition principle to any downwash function $w(\vartheta,s)$; s is the path of the airfoil. Let $w(\vartheta,s) = 0$ for $s < 0$. Then the solution reads:

$$\gamma(\theta,s) = \frac{1}{\pi}\int_0^\pi \left\{\left[(1+\cos\vartheta)\cot\frac{\theta}{2} + \frac{2\sin\theta}{\cos\theta - \cos\vartheta}\right] w(\vartheta,s)\right.$$

$$+\sin\vartheta \ln\frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)}\frac{\partial w(\vartheta,s)}{\partial s}$$

$$\left.+(1-\cos\vartheta)\cot\frac{\theta}{2}\int_0^s U_1(s-\sigma)\frac{\partial w(\vartheta,\sigma)}{\partial\sigma}\,d\sigma\right\} d\vartheta \tag{34}$$

$$U_1(s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{T(-i\omega)}{\omega}\exp \omega s\, d\omega \tag{35}$$

The real function $U_1(s)$ can be represented by series or a pure real integral over tabulated functions (references 11 and 12). Numerical values are given in table 2. It should be noted that (34) holds for variable flying speed too, but (32) for constant flying speed only.

Next we shall analyze the yawing airfoil of infinite span and constant chord. The angle between leading edge and axis y, the so-called angle of yaw, is denoted with δ. The half airfoil chord l is always measured parallel to axis x. Putting $c = \infty$ and substituting

$$x - y\tan\delta = s;\quad \xi - \eta\tan\delta = \sigma$$

gives, from (23), the downwash

$$w(s,0,t) = -\frac{1}{2\pi\cos^2\delta}\int_{\sigma=-l}^{+l}\int_{x''=\sigma}^{\infty}\gamma(\sigma,t)\exp\frac{i\nu}{v}(\sigma-x'')\frac{dx''d\sigma}{(s-x'')^2} \tag{36}$$

This result differs from (31) only in the constant factor $\cos^2\delta$. With given downwash the pressure jump of the yawing airfoil of infinite span is therefore smaller by the factor $\cos^2\delta$ than that of the normally flying airfoil of equal chord measured in direction of flight. This result is synonymous with that, in that only the normal component of the flying speed, i.e., that located in the plane normal to the airfoil, is aerodynamically effective. It is readily apparent that in frictionless fluid the tangential component, which corresponds to a displacement of the airfoil of infinite span along its generating line, must be devoid of aerodynamic effect, and therefore not ascertainable by pressure measurements outside the airfoil.

But, if the pressure is measured in a reference system that is solidly connected with the transversely moving airfoil, the Lorentz transformation is dictated if the fluid is compressible. With v_2 denoting the transverse velocity measured contrary to direction y, it is necessary, according to (5) to substitute

$$t = \frac{t' + v_2 y'/c^2}{\sqrt{1 - v_2^2/c^2}}$$

in the integral (26).

Next comes the case of the airfoil with periodic γ distribution across the span. Let

$$\gamma(\xi,\eta,t) = \gamma(\xi,t) \exp i \mu \eta \tag{37}$$

The study is restricted to the special case $c = \infty$, that is, in compressible fluid. Putting

$$u = \frac{\eta - y}{\sqrt{x'^2 + z^2}}$$

while noting that the part integrals from $+\infty$ to y and from y to $-\infty$ must be of the same magnitude, affords the determinate integral (reference 13)

$$\int_0^\infty \exp(-xu) \frac{du}{\sqrt{u^2 + 1}} = \frac{\pi}{2} [Z_o(x) - N_o(x)] \tag{38}$$

where N is Neumann's cylindrical function and Z the Struve function. Abbreviating

$$\left.\begin{aligned} G_o(x) &= Z_o(i\ x) - N_o(i\ x) \\ G_1(x) &= Z_1(i\ x) - N_1(i\ x) - \frac{2}{\pi} \end{aligned}\right\} \tag{39}$$

$$w(x,y,z,t)=\frac{1}{4}\exp i\mu y\int_{\xi=-l}^{+l}\int_{-\infty}^{x'=x-\xi}dx'd\xi\ \gamma\left(\xi,t+\right.$$

$$\left.\frac{x'-x+\xi}{v}\right)\frac{\partial^2}{\partial z^2}G_0\left(-\mu\sqrt{x'^2+z^2}\right) \tag{40}$$

Since

$$\frac{d\ G_0(x)}{dx}=-i\ G_1(x)$$

the differential followed by insertion of $z = 0$, $x'' = x - x'$ gives the downwash at

$$w(x,y,0,t)=\frac{i\mu}{4}\exp i\mu y\int_{\xi=-l}^{+l}\int_{x''=\xi}^{+\infty}\gamma\left(\xi,t\right.$$

$$\left.+\frac{\xi-x''}{v}\right)G_1(-\mu\,|x-x''|)\frac{dx''\,d\xi}{|x-x''|} \tag{41}$$

Integral equation (41) is similar in structure to (26). Its further integration requires a special assumption for the time function. If harmonic oscillations are involved, a real formula must be used or (41) must be written. Since

$$\lim_{x\to 0} xG_1(x)=-\frac{2i}{\pi} \tag{42}$$

the downwash for $\mu \longrightarrow 0$ follows from (41) and (42) at

$$w(x,0,t)=-\frac{1}{2\pi}\int_{\xi=-l}^{+l}\int_{x''=\xi}^{+\infty}\gamma\left(\xi,t+\frac{\xi-x''}{v}\right)\frac{dx''d\xi}{(x-x'')^2} \tag{43}$$

8. THE AIRFOIL WITH HIGH ASPECT RATIO

According to (23) the downwash of a flat oscillating airfoil in incompressible fluid is

$$w(x,y,z,t) = \frac{1}{4\pi}\iint_{Fl}\int_{-\infty}^{x'=x-\xi} dx'\,d\xi\,d\eta\,\gamma(\xi,\eta,t)\,\exp\frac{i\nu}{v}(x'-x+\xi) \times \frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{x'^2+(y-\eta)^2+z^2}} \tag{44}$$

The introduction of the new integration variable

$$x' = -u\sqrt{(y-\eta)^2+z^2} \tag{45}$$

in (44) affords

$$w(x,y,z,t) = \frac{1}{4\pi}\iint_{Fl} d\xi\,d\eta\,\gamma(\xi,\eta,t)\,\exp\frac{i\nu}{v}(-x+\xi) \times \frac{\partial^2}{\partial z^2}\int_{u=\frac{-x'+\xi}{\sqrt{(y-\eta)^2+z^2}}} \exp\left(-\frac{i\nu}{v}\sqrt{(y-\eta)^2+z^2}\,u\right)\frac{du}{\sqrt{u^2+1}} \tag{46}$$

On an airfoil with high aspect ratio we find for most points of the surface

$$|x-\xi| \ll |y-\eta| \tag{47}$$

Only in the closer proximity of the starting point, which itself may be situated on the airfoil, does inequation (47) fail to hold unconditionally. The more slender the contour the smaller the error introduced when the lower limit of the integral is approximately put at $u = 0$ in (46). Following this, the integration can be carried out according to (38) with known functions, ultimately yielding, after differentiation, the downwash

$$w(x,y,0,t) \approx \frac{-i\nu}{8v}\iint_{Fl} d\xi\,d\eta\,\gamma(\xi,\eta,t)\,\exp\frac{i\nu}{v}(-x+\xi) \times \frac{G_1\left(\frac{\nu}{v}|y-\eta|\right)}{|y-\eta|} \tag{48}$$

Equation (48) is substantially simpler than the original (44). Admittedly, it holds only for special downwash distributions in x direction; namely,

$$w(x,y,0,t) = \overline{w}(y) \exp i\nu\left(t - \frac{x}{v}\right) \tag{49}$$

as is readily seen from a comparison of (48) and (49). It affords the downwash function

$$\overline{w}(y) \approx \frac{-i\nu}{8v}\iint_{F1} \gamma(\xi,\eta) \exp \frac{i\nu\xi}{v}\, G_1\left(\frac{\nu}{v}\,|y-\eta|\right)\frac{d\xi d\eta}{|y-\eta|} \tag{50}$$

This equation can be solved implicitly only with a supplementary assumption for function $\gamma(\xi,\eta)$. As the next assumption it is established that the γ distribution as a function of ξ, up to a factor $c(\eta)$ dependent on η, shall be such as if two-dimensional flow existed in every section. This is the second essential approximate assumption. The downwash being given by (49), the desired γ distribution can be computed from (32) and the integral

$$K(\eta) = \int_{l_2(\eta)}^{l_1(\eta)} \gamma(\xi,\eta) \exp \frac{i\nu\xi}{v}\, d\xi \tag{51}$$

formed. Entering (49) in (32) affords with

$$\xi = \xi_o(\eta) - l(\eta)\cos\vartheta ; \overline{\omega}(\eta) = \nu\, l(\eta)/v$$

$$\gamma(\vartheta,\eta) = C(\eta)\, \overline{w}(\eta) \exp\left(-\frac{i\nu}{v}\xi_o(\eta)\right) \cot\frac{\vartheta}{2}\,[J_o(\overline{\omega}) + i J_1(\overline{\omega}) + (J_o(\overline{\omega}) - i J_1(\overline{\omega}))\, T(\overline{\omega})] \tag{52}$$

$\xi_o(\eta)$ is the coordinate of the airfoil medial line. For airfoil contours that are symmetrical to the η axis, $\xi_o = 0$.

J_o and J_1 are Bessel's functions; T is given by (33) as quotient of Hankel's functions. Entering (52) in (51), followed by integration with respect to ξ, gives

$$\left.\begin{aligned} K(\eta) &= C(\eta)\,\pi\,\overline{w}(\eta)\, l(\eta)\, f(\overline{\omega}(\eta)) \\ f(\overline{\omega}) &= J_o^2(\overline{\omega}) + J_1^2(\overline{\omega}) + (J_o(\overline{\omega}) - i J_1(\overline{\omega}))^2\, T(\overline{\omega}) \end{aligned}\right\} \tag{53}$$

If the flow were actually plane, $C(\eta)$ would be equal to 1.

Next (51) is entered in (50). Since (50), for reasons outlined previously, is divergent, partial integration is necessary for forming Cauchy's principal value. Then

$$S(x) = \frac{i\pi}{2}\, x \int_x^{\infty} G_1(u)\, \frac{du}{u} \tag{54}$$

$$\overline{w}(y) \approx F(S,K) + \frac{1}{4\pi} \int_{-b}^{+b} S\left(\frac{\nu}{v}\left|y - \eta\right|\right) \frac{dK(\eta)}{d\eta}\, \frac{d\eta}{y - \eta} \tag{55}$$

By unlimited increase of aspect ratio, $dK/d\eta$ tends toward zero. Then exact plane flow prevails and

$$\overline{w}(y) = F(S,K) = \frac{K\ (y)}{\pi l(y)\ f(\overline{\omega}(y))} \tag{56}$$

according to (53) and (55). The final form of integral equation for the oscillating airfoil of large aspect ratio follows from (55) and (56) at

$$\overline{w}(y) \approx \frac{K\ (y)}{\pi l(y) f(\overline{\omega}(y))} + \frac{1}{4\pi} \int_{-b}^{+b} S\left(\frac{\nu}{v}\left|y - \eta\right|\right) \frac{dK(\eta)}{d\ \eta}\, \frac{d\eta}{y - \eta} \tag{57}$$

Putting $\nu = 0$,

$$S(0) = 1;\ f(0) = 2$$

gives, from equation (57), the stationary form

$$\overline{w}(y) \approx \frac{K(y)}{2\pi l(y)} + \frac{1}{4\pi} \int_{-b}^{+b} \frac{dK(\eta)}{d\ \eta}\, \frac{d\ \eta}{y - \eta} \tag{58}$$

This is the well-known Prandtl integral equation of "vortex filament theory." One usually substitutes in the first term of (58) for π the measured stationary circulation constant of a real airfoil section $c_1 < \pi$, although the whole theory dealt with here applies only to airfoils. Quantity K in (58) has a very elementary significance; namely,

$$K(\eta) = \int \gamma(\xi,\eta)\, d\xi$$

that is, the so-called circulation of the wing. The significance of K in (57) is less elementary. Cicala (reference 14) and others have attempted to extend the vortex filament theory by means of vortex concepts to the case of the oscillating airfoil and actually arrived at an equation agreeable with (57). But Cicala arbitrarily identifies K with the total circulation (sum of bound and free vortices along wing chord) and accordingly finds

$$\frac{1}{f(\overline{\omega})} = \frac{\pi}{4}\,\overline{\omega}\,\exp i\,\overline{\omega}\,[H_0^{(2)}(\overline{\omega}) - i\,H_1^{(2)}(\overline{\omega})]$$

As concerns the permissible downwash distribution in x direction no data are given at all. So, in this instance, it is necessary to state, the vortex method leads to inaccurate results. Equation (58) cannot be generalized to (57) by vortex theory, although (57) can be specialized in (58).

Equation (57) first gives quantity $K(y)$. But to compute the lift requires the bound circulation

$$\Gamma(\eta) = \int \gamma(\xi,\eta)\,d\xi \qquad (59)$$

Putting (52) in (59), followed by integration with respect to ξ yields

$$\Gamma(\eta) = C(\eta)\,\pi\overline{w}(\eta)\,l(\eta)\,\exp\left(-\frac{i\nu}{v}\,\xi_0(\eta)\right)[J_0(\overline{\omega})$$

$$+ i\,J_1(\overline{\omega}) + (J_0(\overline{\omega}) - i\,J_1(\overline{\omega}))T(\overline{\omega})] \qquad (60)$$

Comparison of (53) and (60) gives the bound circulation

$$\Gamma(y) = \frac{K(y)\,\exp\left(-\frac{i\nu}{v}\,\xi_0(y)\right)}{J_0(\overline{\omega}(y)) - i\,J_1(\overline{\omega}(y))} \qquad (61)$$

It is again pointed out that (57) obtained from (44) by two approximate assumptions applies only to downwash functions of the form (49), where $\overline{w}(y)$ can be a complex function. Other downwash functions cannot be treated by (57) unless one is satisfied to subtract a downwash function w in the form of (49) from the given downwash function w_g and to integrate the smallest possible difference $w_g - w$ on the assumption of two-dimensional flow, by means of (32). This method has been used until the present time in the practical application of the stationary form (58) without being at all clear that it includes a supplementary assumption going beyond the vortex filament theory.

When an attempt is made to apply (57) under the cited assumptions approximately to any other downwash distribution $w(\vartheta,y)$, the question of what mean value to employ for $\overline{w}(y)$ in (57) arises. By assuming plane flow quantity γ can be computed with (32) for any functions $w(\vartheta,y)$. Equation (49) is then so chosen that it affords the same characteristic quantity K on the assumption of plane flow as the given downwash $w(\vartheta,y)$ in plane flow. Then

$$K_1(y)=l(y)\exp\frac{i\nu}{v}\xi_0(y)\int_0^\pi \gamma(\theta,y)\exp(-i\overline{\omega}\cos\theta)\sin\theta\,d\theta$$

$$=\exp\frac{i\nu}{v}\xi_0(y)\frac{l(y)}{\pi}\int_0^\pi\int_0^\pi\left\{[1+\cos\vartheta+(1-\cos\vartheta)T(\overline{\omega})]\cot\frac{\theta}{2}\right.$$

$$\left.+\frac{2\sin}{\cos\theta-\cos\theta}+i\overline{\omega}\sin\vartheta\ \ln\frac{1-\cos(\theta+\vartheta)}{1-\cos(\theta-\vartheta)}\right\}w(\vartheta,y)\times\exp(-i\overline{\omega}\cos\theta)\sin\theta\,d\theta\,d\vartheta$$

$$=l(y)\exp\left(\frac{i\nu}{v}\xi_0(y)\right)\int_0^\pi(1-\cos\vartheta)w(\vartheta,y)\,d\vartheta\,[J_0(\overline{\omega})+iJ_1(\overline{\omega})$$
$$+\left(J_0(\overline{\omega})-iJ_1(\overline{\omega})\right)T(\overline{\omega})] \tag{62}$$

On the other hand, equation (49) gives, according to (53)

$$K_2(y)=\pi l(y)\overline{w}(y)\,[J_0(\overline{\omega})-i\,J_1(\overline{\omega})]$$
$$[J_0(\overline{\omega})+i\,J_1(\overline{\omega})+(J_0(\overline{\omega})-i\,J_1(\overline{\omega}))\,T(\overline{\omega})] \tag{63}$$

Putting $K_1(y)=K_2(y)$, equations (62) and (63) give

$$\overline{w}(y)=\frac{\frac{1}{\pi}\int_0^\pi(1-\cos\vartheta)w(\vartheta,y)\,d\vartheta}{J_0(\overline{\omega})-i\,J_1(\overline{\omega})}\exp\frac{i\nu}{v}\,\xi_0(y) \tag{64}$$

The numerator of this fraction represents the known integral term w_0 that is decisive for the circulation in plane flow. By translational and rotary oscillations of a flat airfoil, w_0 is equal to the downwash at 3/4 wing chord (rear neutral point). For stationary flow the denominator of the fraction (64) is equal to 1; then $\overline{w} = w_0$. But this does not hold for nonstationary flow. Putting, as a check, (49) in (64) identically satisfies (64).

Allowing for (53), (61), and (64), equation (57) can be written in the form

$$\frac{1}{\pi}\int_0^{\pi}(1-\cos\vartheta)w(\vartheta,y)d\vartheta = \frac{\Gamma(y)}{\pi l(y)\left[\frac{J_0+iJ_1}{J_0-iJ_1} - \frac{H_0{}^{(2)}+i\,H_1{}^{(2)}}{H_0{}^{(2)}-i\,H_1{}^{(2)}}\right]}$$

$$+ \exp\left(-\frac{i\nu}{v}\,\xi_0(y)\right)\frac{J_0-iJ_1}{4\pi}\int_{-b}^{+b}S\left(\frac{\nu}{v}\left|y-\eta\right|\right)$$

$$\frac{d}{d\eta}\left[\Gamma(\eta)\,(J_0-iJ_1)\,\exp\frac{i\nu}{v}\,\xi_0(\eta)\right]\frac{d\eta}{y-\eta} \tag{65}$$

where J and H are functions of $\overline{\omega}(y)$ outside the integral and of $\overline{\omega}(\eta)$ inside of the integral; Γ is the customary bound circulation. In consequence of the approximate assumption regarding the mean value of $\overline{w}$, equation (65) no longer gives the circulation of the plane flow exact for infinite aspect ratio, which makes it neessary to put the first quotient of the first brace equal to one. This is apparently as justified as in equation (56).

Visualizing a stationary periodic downwash field, while discounting for the time being the effect of the vortices already contained in this velocity field on the airfoil, a flat airfoil moving rectilinearly and at constant speed through this downwash field, is continually subject to a downwash of the form (49), whence (57) is forthwith applicable. Any stationary downwash fields can be presented by linear superposition of several such fields of different periods. Equation (57) is therefore especially suitable for treating gust stresses of airfoils with large aspect ratio.

The complex function $S(x)$ occurring in the kernel of the integral (57) is given by (39) and (54). Numerical values are given in table 3. Because of

$$\lim_{x \to \infty} x^2\,G_1(x) = -\,2^{*} \tag{66}$$

*Probably an error.

$$\lim_{x \to \infty} x\, S(x) = -\frac{i}{2}$$

follows from (54) and (66).

In rapid oscillations, that is, large values of $\frac{\nu}{v}\,|y - \eta|$, the induction effect given by the second term of (57) is therefore very small, and in the extreme case of high frequency an almost plane flow must be counted on even on an airfoil of finite span, if the aspect ratio is high. This is an important result.

Theoretically the solution of (57) can be effected in the same manner as the much discussed equation (58), although the addition of the complex function S makes it more protracted. The result, which is, moreover, usually encumbered by the assumption (64) in partial cases, consists on airfoils with large aspect ratio and at the practical values of $\overline{\omega}$ in the order of magnitude of 1.0 only in a small correction relative to plane flow.

In case such refinements of the solution are not deemed necessary, it is more appropriate to apply an iteration method which ties in with the exact integral equation (44). A somewhat correct approximate solution $\gamma_1(\xi,\eta,t)$ for a given downwash w is afforded from (32) on the premise of plane flow. Entering γ_1 in (44) gives a downwash $w_1 \neq w$. The difference $w - w_1$ is then entered again in (32); $\Delta\gamma$ is computed and this added to γ as correction factor, etc. The convergence of this method needs to be checked of course from one case to the other, although it should be sufficient in general for slender airfoils.

There is nothing to prohibit the application of this method to compressible fluid $(\beta \neq 0)$, once the general solution of (26) in form of an integral representation of the type of solution (32) found for $\beta = 0$, is available.

9. SYSTEMS OF SEVERAL AIRFOILS

The general form (19) of the integral equation of the airfoil theory comprises the possibility that the surface integral $d\xi d\eta$ can be extended over several spatially separated regions of the airfoil. A case in point is the biplane or the wing with split flap. Of course, Kutta's flow-off condition must be satisfied for the trailing edges of each part of the airfoil.

Given a general solution $\gamma = \underline{L}(w)$ of the integral equation for a single airfoil, the problem of two airfoils can be solved by successive approximation according to the superposition principle. Assume the given downwash of the airfoil (1) as $w_g^{(1)}$ and of airfoil (2) as $w_g^{(2)} = 0$. First compute as solution in first approximation

$$\gamma^{(1)} = \underline{L}^{(1)}(w_g^{(1)})$$

As a result of this pressure distribution, there is in conformity with the general integral equation an always computable downwash field

$$w^{(2)} = \underline{J}(\gamma^{(1)})$$

which induces on airfoil (2) the pressure distribution

$$\Delta\gamma^{(2)} = \underline{L}^{(2)}(w^{(2)})$$

which in turn creates the downwash field

$$\Delta w^{(1)} = \underline{J}(\Delta\gamma^{(2)})$$

which induces on airfoil (1) the pressure distribution

$$\Delta\gamma^{(1)} = \underline{L}^{(1)}(\Delta w^{(1)})$$

In the second approximation the pressure distribution of airfoil (1) is therefore given by $\gamma^{(1)} + \Delta\gamma^{(1)}$, and that of airfoil (2) by $\Delta\gamma^{(2)}$. This method of iteration can then be continued. In cases where the distance of the airfoils is not very small in relation to the wing chord, it converges very well, as proved by Kleinwächter (reference 15) for stationary flow.

The iteration method is chiefly opportune when the solution $\underline{L}$ in continuous form is known. Thus for the present it is restricted to two-dimensional flow with $c = \infty$, for which the solutions (32) and (34), respectively, are available.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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Table I. Function $T(\overline{\omega})$.

$\overline{\omega}$	Re T	Jm T	$\overline{\omega}$	Re T	Jm T
0,000	1,00000	0,00000	0,72	0,12481	— 0,24865
0,002	0,99342	— 0,02516	0,74	0,12034	— 0,24458
0,010	0,96525	— 0,09135	0,76	0,11611	— 0,24061
0,02	0,92745	— 0,15042	0,78	0,11210	— 0,23676
0,04	0,85340	— 0,23200	0,80	0,10829	— 0,23300
0,06	0,78408	— 0,28519	0,82	0,10467	— 0,22935
0,08	0,72086	— 0,32080	0,84	0,10123	— 0,22580
0,10	0,66385	— 0,34460	0,86	0,09795	— 0,22234
0,12	0,61265	— 0,36015	0,88	0,09483	— 0,21898
0,14	0,56674	— 0,36978	0,90	0,09186	— 0,21570
0,16	0,52554	— 0,37513	0,92	0,08902	— 0,21251
0,18	0,48851	— 0,37735	0,94	0,08631	— 0,20940
0,20	0,45516	— 0,37725	0,96	0,08372	— 0,20637
0,22	0,42504	— 0,37545	0,98	0,08124	— 0,20342
0,24	0,39778	— 0,37239	1,00	0,07887	— 0,20055
0,26	0,37303	— 0,36841	1,1	0,06843	— 0,18721
0,28	0,35050	— 0,36376	1,2	0,05991	— 0,17542
0,30	0,32994	— 0,35864	1,3	0,05287	— 0,16493
0,32	0,31114	— 0,35319	1,4	0,04699	— 0,15555
0,34	0,29389	— 0,34752	1,5	0,04203	— 0,14713
0,36	0,27804	— 0,34172	1,6	0,03780	— 0,13953
0,38	0,26344	— 0,33585	1,7	0,03417	— 0,13264
0,40	0,24995	— 0,32997	1,8	0,03103	— 0,12636
0,42	0,23748	— 0,32411	1,9	0,02830	— 0,12063
0,44	0,22592	— 0,31831	2,0	0,02591	— 0,11538
0,46	0,21518	— 0,31258	2,5	0,01749	— 0,09459
0,48	0,20518	— 0,30695	3,0	0,01256	— 0,08001
0,50	0,19587	— 0,30142	3,5	0,00944	— 0,06924
0,52	0,18718	— 0,29600	4,0	0,00734	— 0,06099
0,54	0,17905	— 0,29071	4,5	0,00587	— 0,05447
0,56	0,17144	— 0,28554	5,0	0,00479	— 0,04920
0,58	0,16430	— 0,28049	10,0	0,00124	— 0,02489
0,60	0,15760	— 0,27557	20	0,00031	— 0,01249
0,62	0,15130	— 0,27078	30	0,00014	— 0,00833
0,64	0,14537	— 0,26611	40	0,00008	— 0,00625
0,66	0,13978	— 0,26156	50	0,00005	— 0,00500
0,68	0,13450	— 0,25714	100	0,00001	— 0,00250
0,70	0,12952	— 0,25284	∞	0	0

Table II. Function $U_1(s)$.

s	$U_1(s)$	s	$U_1(s)$
0,0	0,0000	3,0	0,4391
0,1	0,0244	3,5	0,4799
0,2	0,0476	4,0	0,5159
0,3	0,0698	4,5	0,5479
0,4	0,0910	5,0	0,5764
0,5	0,1113	5,5	0,6020
0,6	0,1308	6,0	0,6251
0,7	0,1494	6,5	0,6460
0,8	0,1674	7,0	0,6650
0,9	0,1846	7,5	0,6824
1,0	0,2012	8,0	0,6983
1,1	0,2172	8,5	0,7128
1,2	0,2326	9,0	0,7263
1,3	0,2475	9,5	0,7386
1,4	0,2618	10	0,7501
1,5	0,2757	11	0,7706
1,6	0,2891	12	0,7883
1,7	0,3021	15	0,8296
1,8	0,3146	20	0,8733
1,9	0,3268	25	0,9003
2,0	9,3386	30	0,9183
2,1	0,3500	40	0,9405
2,2	0,3611	50	0,9535
2,3	0,3719	100	0,9781
2,4	0,3823	500	0,9959
2,5	0,3924	1000	0,9980
		∞	1,0000

Table III. Function $S(x)$.

x	Re $S(x)$	Jm $S(x)$	x	Re $S(x)$	Jm $S(x)$
0,00	1,00000	0,00000	1,0	0,27362	— 0,37626
0,02	0,96969	— 0,07315	1,2	0,21564	— 0,35590
0,04	0,94104	— 0,11919	1,4	0,17050	— 0,33352
0,06	0,91373	— 0,15537	1,6	0,13516	— 0,31091
0,08	0,88759	— 0,18537	1,8	0,10739	— 0,28905
0,10	0,86252	— 0,21091	2,0	0,08549	— 0,26845
0,12	0,83842	— 0,23302	2,2	0,06817	— 0,24933
0,14	0,81522	— 0,25235	2,4	0,05444	— 0,23177
0,16	0,79286	— 0,26940	2,6	0,04353	— 0,21574
0,18	0,77129	— 0,28451	2,8	0,03485	— 0,20118
0,20	0,75046	— 0,29795	3,0	0,02792	— 0,18798
0,22	0,73033	— 0,30994	3,2	0,02240	— 0,17603
0,24	0,71087	— 0,32066	3,4	0,01798	— 0,16521
0,26	0,69203	— 0,33025	3,6	0,01445	— 0,15542
0,28	0,67380	— 0,33884	3,8	0,01162	— 0,14655
0,30	0,65615	— 0,34652	4,0	0,00935	— 0,13851
0,32	0,63904	— 0,35339	4,2	0,00753	— 0,13120
0,34	0,62246	— 0,35953	4,4	0,00606	— 0,12455
0,36	0,60638	— 0,36499	4,6	0,00489	— 0,11849
0,38	0,59079	— 0,36985	4,8	0,00394	— 0,11295
0,40	0,57566	— 0,37415	5,0	0,00318	— 0,10787
0,42	0,56098	— 0,37794	5,2	0,00257	— 0,10322
0,44	0,54672	— 0,38127	5,4	0,00207	— 0,09894
0,46	0,53288	— 0,38416	5,6	0,00167	— 0,09499
0,48	0,51943	— 0,38666	5,8	0,00135	— 0,09134
0,50	0,50637	— 0,38879	6,0	0,00109	— 0,08796
0,52	0,49368	— 0,39059	7,0	0,00038	— 0,07425
0,54	0,48135	— 0,39207	8,0	0,00013	— 0,06431
0,56	0,46936	— 0,39326	9,0	0,00005	— 0,05677
0,58	0,45770	— 0,39419	10,0	0,00002	— 0,05084
0,60	0,44637	— 0,39487	12,0	0,00000	— 0,04214
0,70	0,39416	— 0,39511	14,0	0,00000	— 0,03602
0,80	0,34858	— 0,39132	20,0	0,00000	— 0,02510
0,90	0,30866	— 0,38475	∞	0	0